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THE ROLE OF DILATIONS IN DIFFEOMORPHISM COVARIANT ALGEBRAIC QUANTUM FIELD THEORY *

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Abstract

The quantum analogue of general relativistic geometry should be implementable on smooth manifolds without an a priori metric structure, the kinematical covariance group acting by diffeomorphisms.

Here I approach quantum gravity (QG) in the view of constructive, algebraic quantum field theory (QFT). Comparing QG with usual QFT, the algebraic approach clarifies analogies and peculiarities. As usual, an isotonic net of $*$ -algebras is taken to encode the quantum field operators. For QG, the kinematical covariance group acts via diffeomorphisms on the open sets of the manifold, and via algebraic isomorphisms on the algebras. In general, the algebra of observables is covariant only under a (dynamical) subgroup of the general diffeomorphism group.

After an algebraic implementation of the dynamical subgroup of dilations, small and large scale cutoffs may be introduced algebraically. So the usual a priori conflict of cutoffs with general covariance is avoided. Even more, these cutoffs provide a natural local cobordism for topological quantum field theory.

A new commutant duality between the minimal and maximal algebra allows to extract the modular structure from the net of algebras. The outer modular isomorphisms are then again related to dilations, which (under certain conditions) may provide a notion of time.

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1 Introduction

The following investigations might be seen as an attempt to understand some aspects of quantum field theory (QFT) on differentiable manifolds. This is indeed a very promising approach to quantum general relativity[1].

The observation procedures represent the abstract kinematical framework for possible preparations of measurements, while the observables encode the kinds of questions one can ask from the physical system. The covariance group of the observation procedures reflects a general (a priori) redundancy of their mathematical implementation. The more sophisticated the structure of the observation procedures, the smaller the covariance group will be in general. In a concrete observation the kinematical covariance will be broken. So, e.g., in a treelike network of string world tubes a concrete local observation requires the explicit selection of one of many a priori equivalent vertices, whence it breaks the covariance which holds for the network of vertices as a whole[2]. However, irrespectively of the loss of covariance in a concrete observation, the action of the covariance group may still be well defined on the observation procedures. In any case, the loss of covariance in a concrete observation is related to a specific structure of the state of the physical system.

Previous attempts[3, 4] to implement kinematical general covariance and its dynamical breakdown in the spirit of an algebraic, constructive approach[5] to quantum (field) theory have been continued recently[6, 7, 8]. The principle of locality is kept by demanding that, observation procedures correspond to possible preparations of localized measurements in bounded regions. Note that here is no a priori notion of neither a metric, nor a time, nor even a causal structure. Then, on different regions there will be no a priori causal relations between observables.

For a net of subalgebras of a Weyl algebra, it is indeed possible[9] to work with a flexible notion of causality rather than with a rigidly given one. In principle it might be possible to construct the net together with its underlying manifold from a partial order via inclusion of the algebras themselves only[10].

Nevertheless, below we start just from a net of $*$ -algebras on a differentiable manifold. On this net, a physical state induces dynamical relations, whence the algebra of observables is covariant just under a certain subgroup of the general diffeomorphism group. This subgroup describes a covariance related to the (dynamically relevant) observables. The present examinations emphasize on the dynamical subgroup of dilations.

For QFT on Minkowski space (see Sect. 3 below) the usual causality condition can be sharpened to Haag duality[11], which there in particular opens the door for the powerful DHR-analysis[12], Haag duality is the first example of an algebraic commutant duality on a net of von Neumann algebras. Recently[6, 7, 8], attempts were made to find an appropriate modification of Haag duality that can be applied in consistency with general covariance on an arbitrarily curved background. In Sect. 4 below, the idea of a certain algebraic commutant duality between very large scales and very small scales is exploited, in order to obtain a modular group of algebraic isomorphisms on a net of von Neumann algebras.

In [6] such an idea was implemented for the asymptotical large scale limit $n \rightarrow \infty$ of discretely parametrized domains \mathcal{O}_n , $n \in \mathbb{N}$.

Unlike there, in [7, 8] a von Neumann commutant duality $\mathcal{R}(\mathcal{O}_{s_{\min}}^x) = \mathcal{R}'(\mathcal{O}_{s_{\max}}^x)$ was

introduced between some minimal and some maximal algebra from a net of von Neumann algebras on continuously parametrized bounded domains \mathcal{O}_s^x , $s_{\min} \leq s \leq s_{\max}$, around any point x .

Different than in [6], it is considered here to be more natural that both, $\mathcal{O}_{s_{\min}}^x$ and $\mathcal{O}_{s_{\max}}^x$ are nonempty bounded open sets, with their boundaries representing some kind of horizon for observers located outside $\mathcal{O}_{s_{\max}}^x$ and inside $\mathcal{O}_{s_{\min}}^x$ respectively. These bounded sets simultaneously introduce a small and large scale regularization naturally from the commutant duality for isotonic algebras. In this approach we avoid to fall in an a priori conflict like between the usual cutoffs and (general) covariance.

In Sect. 5 the commutant duality, together with the isotony property, is used to extract the modular structure. The latter is related to local dilations on the net.

Sect. 6 concludes with a brief discussion of some possible implications of the proposed structure for quantum general relativity and a posteriori notions of time and causality.

2 Algebraic axioms of QFT on Minkowski space

Usual QFT on Minkowski space \mathcal{M} may be formulated in terms of a net of von Neumann algebras $\mathcal{R}(\mathcal{O})$ on open sets $\mathcal{O} \subset \mathcal{M}$, satisfying the following axioms:

Isotony

$$\mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2) . \quad (2.1)$$

Additivity

$$\mathcal{O} = \cup_j \mathcal{O}_j \quad \Rightarrow \quad \mathcal{R}(\mathcal{O}) = (\cup_j \mathcal{R}(\mathcal{O}_j))'' . \quad (2.2)$$

Causality

$$\mathcal{O}_1 \perp \mathcal{O}_2 \quad \Rightarrow \quad \mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2)' . \quad (2.3)$$

Covariance

$$P \ni g \mapsto U(g) \in U(P) \quad : \quad \mathcal{R}(g\mathcal{O}) = U(g)\mathcal{R}(\mathcal{O})U(g)^{-1} . \quad (2.4)$$

Spectrum Condition

$$\text{spec} U(\tau) \subset \overline{V^+} , \quad \tau \subset P . \quad (2.5)$$

Vacuum Vector

$$\begin{aligned} \exists \Omega \in \mathcal{H}, \|\Omega\| = 1 \quad : \\ \begin{aligned} & \text{(cyclic)} && (\cup_{\mathcal{O}} \mathcal{R}(\mathcal{O})) \Omega \stackrel{\text{dense}}{\subset} \mathcal{H} \\ & \text{(invariant)} && U(g)\Omega = \Omega , \quad g \in P . \end{aligned} \end{aligned} \quad (2.6)$$

In the case of an arbitrarily curved manifold M , condition (2.1) will be kept unmodified, an analogue of condition (2.4) will be maintained, (2.3) will not survive in this way, instead a new commutant duality will be introduced on the boundary of the net, (2.5) does no longer makes sense here, and a cyclic vector like in (2.6) will no longer be unique but state dependent, while its gauge invariance will survive in modified form. The present investigations will not solve the problem, how to relate algebras on arbitrarily intersecting sets, whence (2.2) will be just ignored here.

3 Covariant nets of algebras

Given a differentiable manifold M , a collection $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in M}$ of $*$ -algebras $\mathcal{A}(\mathcal{O})$ on bounded open sets $\mathcal{O} \in M$ is called a *net of $*$ -algebras*, iff

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) . \quad (3.1)$$

The net is sometimes also denoted by $\mathcal{A} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$. Selfadjoint elements of $\mathcal{A}(\mathcal{O})$ may be interpreted as possible measurements in \mathcal{O} .

A net of algebras on M is $\text{Diff}(M)$ covariant, if it reflects the $\text{Diff}(M)$ covariance of the underlying manifold M . $\text{Diff}(M)$ then acts by algebraic isomorphisms on $\mathcal{A} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$, i.e. each diffeomorphism $\chi \in \text{Diff}(M)$ induces an algebraic isomorphism α_χ such that

$$\alpha_\chi(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\chi(\mathcal{O})) . \quad (3.2)$$

Two sets \mathcal{O}_1 and \mathcal{O}_2 , related by a topological isomorphism (e.g. a diffeomorphism) χ such that $\chi(\mathcal{O}_1) = \mathcal{O}_2$, may be identified straightforwardly only if there are no further obstructing relations between them. A relation like $\mathcal{O}_1 \subset \mathcal{O}_2$, in addition to the previous one, implies that \mathcal{O}_1 and \mathcal{O}_2 have to be considered as topologically isomorphic, though non-identical, sets. A similar situation holds on the level of algebras. Isotony (3.1) in connection with covariance (3.2) implies that $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ are isomorphic but non-identical algebras. Therefore it was a misleading abuse of terminology in previous papers[7, 8] to call α_χ an algebraic automorphism (as e.g. [11]) although the situation is more complicated in general. In the following, algebras related simultaneously by isotonic inclusion and an algebraic isomorphism are, more correctly, called just isomorphic rather than automorphic algebras.

The state of a physical system is mathematical described by a positive linear functional ω on \mathcal{A} . Given the state ω , one gets via the GNS construction a representation π^ω of \mathcal{A} by a net of operator algebras on a Hilbert space \mathcal{H}^ω with a cyclic vector $\Omega^\omega \in \mathcal{H}^\omega$. The GNS representation $(\pi^\omega, \mathcal{H}^\omega, \Omega^\omega)$ of any state ω has an associated folium \mathcal{F}^ω , given as the family of those states $\omega_\rho := \text{tr} \rho \pi^\omega$ which are defined by positive trace class operators ρ on \mathcal{H}^ω .

Once a physical state ω (which implicitly contains all peculiarities of a particular observation) has been specified, one can consider in each algebra $\mathcal{A}(\mathcal{O})$ the equivalence relation

$$A \sim B \quad : \Leftrightarrow \quad \omega'(A - B) = 0, \quad \forall \omega' \in \mathcal{F}^\omega . \quad (3.3)$$

These equivalence relations generate a two-sided ideal

$$\mathcal{I}^\omega(\mathcal{O}) := \{A \in \mathcal{A}(\mathcal{O}) | \omega'(A) = 0\} \quad (3.4)$$

in $\mathcal{A}(\mathcal{O})$. The (dynamically relevant) state dependent algebra of observables $\mathcal{A}^\omega(\mathcal{O}) := \pi^\omega(\mathcal{A}(\mathcal{O}))$ may be constructed from the (kinematically relevant) algebra of observation procedures $\mathcal{A}(\mathcal{O})$ by taking the quotient

$$\mathcal{A}^\omega(\mathcal{O}) = \mathcal{A}(\mathcal{O})/\mathcal{I}^\omega(\mathcal{O}) . \quad (3.5)$$

The net of state-dependent algebras then is also denoted as $\mathcal{A}^\omega := \bigcup_{\mathcal{O}} \mathcal{A}^\omega(\mathcal{O})$. By construction, any diffeomorphism $\chi \in \text{Diff}(M)$ induces an algebraic isomorphism α_χ of the observation procedures. Nevertheless, for a given state ω , the action of α_χ will in general *not* leave \mathcal{A}^ω invariant. In order to satisfy

$$\alpha_\chi(\mathcal{A}^\omega(\mathcal{O})) = \mathcal{A}^\omega(\chi(\mathcal{O})) . \quad (3.6)$$

the ideal $\mathcal{I}^\omega(\mathcal{O})$ must transform covariantly, i.e. the diffeomorphism χ must satisfy the condition

$$\alpha_\chi(\mathcal{I}^\omega(\mathcal{O})) = \mathcal{I}^\omega(\chi(\mathcal{O})) \quad (3.7)$$

for some algebraic isomorphism α_χ . Due to non-trivial constraints (3.7), the (dynamical) algebra of observables, constructed with respect to the folium \mathcal{F}^ω , does in general no longer exhibit the full $\text{Diff}(M)$ symmetry of the (kinematical) observation procedures. The symmetry of the observables is dependent on (the folium of) the state ω . Therefore, the selection of a folium of states \mathcal{F}^ω , induced by the actual choice of a state ω , results immediately in a breaking of the $\text{Diff}(M)$ symmetry. The diffeomorphisms which satisfy the constraint condition (3.7) form a subgroup. This effective symmetry group is called the *dynamical group* of the state ω . α_χ is called a *dynamical* isomorphism (w.r.t. the given state ω) w.r.t. χ , if (3.7) is satisfied.

The remaining dynamical symmetry group, depending on the folium \mathcal{F}^ω of states related to ω , has two main aspects which we have to examine in order to specify the physically admissible states: Firstly, it is necessary to specify its state dependent algebraic action on the net of observables. Secondly, one has to find a geometric interpretation for the group and its action on M .

If we consider the dynamical group as an *inertial*, and therefore global, manifestation of dynamically ascertainable properties of observables, then its (local) action should be correlated with (global) operations on the whole net of observables. This implies that at least some of the dynamical isomorphisms α_χ are not inner. (For the case of causal nets of algebras it was actually already shown that, under some additional assumptions, the isomorphisms of the algebras are in general not inner[13].)

Note that one might consider instead of the net of observables $\mathcal{A}^\omega(\mathcal{O})$ the net of associated von Neumann algebras $\mathcal{R}^\omega(\mathcal{O})$, which can be defined even for unbounded $\mathcal{A}^\omega(\mathcal{O})$, if we take from the modulus of the von Neumann closure $(\mathcal{A}^\omega(\mathcal{O}))''$ all its spectral projections[3]. Then the isotony (3.1) induces a likewise isotony of the net $\mathcal{R}^\omega := \bigcup_{\mathcal{O}} \mathcal{R}^\omega(\mathcal{O})$ of von Neumann algebras.

4 Local dilations

In the following I want to exhibit a possibility to introduce simultaneously small and large scale cutoff regularizations on the net of von Neumann algebras. This essentially exploits a local partial ordering on the net, which is induced by the isotony property.

Let us now make use of the given (C^∞) topological structure of M and choose at a given point $x \in M$ a topological basis of nonzero open sets $\mathcal{O}_s^x \ni x$, parametrized by a real parameter s with $0 < s < \infty$, such that

$$s_1 < s_2 \Leftrightarrow \text{cl}(\mathcal{O}_{s_1}^x) \subset \mathcal{O}_{s_2}^x \quad (4.1)$$

and

$$s \rightarrow 0 \Leftrightarrow \text{cl}(\mathcal{O}_s^x) \rightarrow \{x\} . \quad (4.2)$$

The standard inclusion $\mathcal{O}_{s_1}^x \subset \mathcal{O}_{s_2}^x$ (used previously[7]) does not exclude the possibility that $\partial\mathcal{O}_{s_1}^x \cap \partial\mathcal{O}_{s_2}^x \neq \emptyset$. Since, for the following, this would be slightly pathological, condition (4.1) uses here $\text{cl}(\mathcal{O}_{s_1}^x) \subset \mathcal{O}_{s_2}^x$ as a slightly more strict inclusion instead.

Let the parameter s be restricted by $0 < s_{\min,x} < s < s_{\max,x} < \infty$. Exploiting local reparametrization invariance, one may assume

$$s_{\min,x} = s_{\min}, \quad s_{\max,x} = s_{\max} \quad \forall x \in M , \quad (4.3)$$

without loss of generality. Then, for each $x \in M$, open sets \mathcal{O}_s^x with $s \in]s_{\min}, s_{\max}[$ generate local cobordisms between $\partial\mathcal{O}_{s_{\min}}^x$ and $\partial\mathcal{O}_{s_{\max}}^x$, and the isotony property (3.1) implies that

$$s_{\min} < s_1 < s_2 < s_{\max} \Rightarrow \mathcal{R}^\omega(\mathcal{O}_{s_{\min}}^x) \subset \mathcal{R}^\omega(\mathcal{O}_{s_1}^x) \subset \mathcal{R}^\omega(\mathcal{O}_{s_2}^x) \subset \mathcal{R}^\omega(\mathcal{O}_{s_{\max}}^x) . \quad (4.4)$$

Here, any diffeomorphism $\mathcal{O}_{s_1}^x \mapsto \mathcal{O}_{s_2}^x$ is a *local dilation* at $x \in M$ from $\mathcal{O}_{s_1}^x$ to $\mathcal{O}_{s_2}^x$. Note that all local dilations which preserve covariance of (4.4) must leave invariant \mathcal{O}_{\min}^x and \mathcal{O}_{\max}^x .

Now, a commutant duality relation between the inductive limits given by the minimal and maximal algebras is introduced,

$$\mathcal{R}^\omega(\mathcal{O}_{s_{\min}}^x) = \left(\mathcal{R}^\omega(\mathcal{O}_{s_{\max}}^x) \right)' , \quad (4.5)$$

where \mathcal{R}' denotes the commutant of \mathcal{R} within some $\mathcal{R}_{\max} \supset \mathcal{R}$. Then the bicommutant theorem ($\mathcal{R}'' = \mathcal{R}$) implies that likewise also

$$\mathcal{R}^\omega(\mathcal{O}_{s_{\max}}^x) = \left(\mathcal{R}^\omega(\mathcal{O}_{s_{\min}}^x) \right)' . \quad (4.6)$$

If one now demands that all maximal (or all minimal) algebras are isomorphic to each other, independently of the choice of x and the open set $\mathcal{O}_{s_{\max}}^x$ (resp. $\mathcal{O}_{s_{\min}}^x$), then by (4.5) (resp. (4.6)) also all minimal (resp. maximal) algebras are isomorphic to each other. The isomorphism class is then an abstract universal minimal resp. maximal algebra, denoted by $\mathcal{R}_{\min}^\omega$ and $\mathcal{R}_{\max}^\omega$ respectively.

If, like in the following, the commutant is always taken within $\mathcal{R}_{\max}^\omega$. the duality (4.5) implies that $\mathcal{R}_{\min}^\omega$ is Abelian. (Note however, that one should also keep in mind the possibility to take the commutant w.r.t. to some larger algebra $\mathcal{R}_B^\omega \supset \mathcal{R}_{\max}^\omega$. Such a choice would possibly include further correlations outside the observable range; it is not considered further here.)

By isotony and (4.1) together with (4.3), the mere existence of $\mathcal{R}_{\min}^\omega$ resp. $\mathcal{R}_{\max}^\omega$ implies the existence of nontrivial sets $\mathcal{O}_{s_{\min}}^x$ resp. $\mathcal{O}_{s_{\max}}^x$ at any $x \in M$. By (4.3) we already gauged the size of all these sets to s_{\min} resp. s_{\max} , i.e. to a common size (as measured by the parameter s) of independently of $x \in M$. So in this case s_{\min} and s_{\max} really denote an universal small resp. large scale cutoff. Note that, in the context of Sect. 3, the universality assumption (4.3) is indeed nontrivial, because local diffeomorphisms consistent with the structure above must preserve s_{\min} , s_{\max} , and the monotony of the ordered set $]s_{\min}, s_{\max}[$. The number $s \in]s_{\min}, s_{\max}[$ parametrizes the partial order of the net of algebras spanned between the inductive limits $\mathcal{R}_{\min}^\omega$ and $\mathcal{R}_{\max}^\omega$. (Strictly speaking, these inductive limits are not part of the diffeorphism invariant net itself !)

Although in local QFT usually the support of an algebra and that of its commutant are not at all related, it might be nevertheless instructive to consider the case where (sufficiently large) algebras of the net satisfy

$$(\mathcal{R}^\omega(\mathcal{O}_s^x))' \subset \mathcal{R}^\omega(\mathcal{O}_s^x) . \quad (4.7)$$

Then, with the center of $\mathcal{R}^\omega(\mathcal{O}_s^x)$ defined as $\mathcal{Z}(\mathcal{R}^\omega(\mathcal{O}_s^x)) := \mathcal{R}^\omega(\mathcal{O}_s^x) \cap (\mathcal{R}^\omega(\mathcal{O}_s^x))'$, one obtains $\mathcal{Z}(\mathcal{R}^\omega(\mathcal{O}_s^x)) = (\mathcal{R}^\omega(\mathcal{O}_s^x))' = \mathcal{Z}((\mathcal{R}^\omega(\mathcal{O}_s^x))')$, and especially $\mathcal{Z}(\mathcal{R}_{\max}^\omega) = \mathcal{R}_{\min}^\omega = \mathcal{Z}(\mathcal{R}_{\min}^\omega)$. So, for a pair of commutant dual algebras satisfying Eq. (4.7), the smaller one is always Abelian, namely it is the center of the bigger one. With (4.7), the isotony of the net implies the existence of an algebra \mathcal{Z}^ω which is *maximal Abelian*, in other words commutant selfdual, satisfying $\mathcal{Z}^\omega = (\mathcal{Z}^\omega)' = \mathcal{Z}(\mathcal{Z}^\omega)$. This algebra is given explicitly via the Abelian net of all centers, $\mathcal{Z}^\omega := \bigcup_{\mathcal{O}} \mathcal{Z}(\mathcal{R}^\omega(\mathcal{O}))$. \mathcal{Z}^ω , located on an underlying set $\mathcal{O}_{s_z}^x$ of intermediate size s.th. $s_{\min} < s_z < s_{\max}$, separates the small Abelian algebras $\mathcal{R}^\omega(\mathcal{O}_s^x) = \mathcal{Z}(\mathcal{R}^\omega(\mathcal{O}_s^x))$, with $s \leq s_z$, from larger non-Abelian algebras $\mathcal{R}^\omega(\mathcal{O}_s^x) = (\mathcal{Z}(\mathcal{R}^\omega(\mathcal{O}_s^x)))'$, with $s > s_z$.

For a net subject to (4.7), its lower end is Abelian, whence observations on small regions with $s \leq s_z$ are expected to be rather classical. Nevertheless, for increasing size $s > s_z$, there might well exist a non-trivial quantum (field) theory (in fact it was shown[13] that, for causal nets, the algebras of QFT are not Abelian and not finite-dimensional). For quantum general relativity there might indeed be a kinetical substructure[14], with classical elementary constituents spanning an Abelian algebra. It is interesting in this context that the Abelian algebra of free loops in quantum general relativity provides indeed such classical constituents[1, 15].

Nevertheless, the following investigations all hold independently from relation (4.7). Indeed we will see below, that (4.7) can only make sense if we take the commutant w.r.t. some algebra essentially larger than $\mathcal{R}_{\max}^\omega$.

5 Modular structure and dilations

If we consider the small and large scale cutoffs as introduced above, it should be clear that only regions of size $s \in]s_{\min}, s_{\max}[$ are admissible for measurement. The commutant duality between $\mathcal{R}_{\min}^\omega$ and $\mathcal{R}_{\max}^\omega$ inevitably yields large scale correlations in the structure of any physical state ω on any admissible region \mathcal{O}_s^x of measurement at x . Let us assume here that ω is properly correlated, i.e. the GNS vector Ω^ω is already cyclic under $\mathcal{R}_{\min}^\omega$.

Then, by duality, it is separating for $\mathcal{R}_{\max}^\omega = \mathcal{R}_{\min}^{\omega'}$. Furthermore Ω^ω is also cyclic under $\mathcal{R}_{\max}^\omega$, and hence separating for $\mathcal{R}_{\min}^\omega$.

So Ω^ω is a cyclic and separating vector for $\mathcal{R}_{\min}^\omega$ and $\mathcal{R}_{\max}^\omega$, and by isotony also for any local von Neumann algebra $\mathcal{R}^\omega(\mathcal{O}_s^x)$.

As a further consequence, on any region \mathcal{O}_s^x , the Tomita operator S and its conjugate F can be defined densely by

$$SA\Omega^\omega := A^*\Omega^\omega \quad \text{for } A \in \mathcal{R}^\omega(\mathcal{O}_s^x), \quad (5.1)$$

$$FB\Omega^\omega := B^*\Omega^\omega \quad \text{for } B \in \mathcal{R}^\omega(\mathcal{O}_s^x)'. \quad (5.2)$$

The closed Tomita operator S has a polar decomposition

$$S = J\Delta^{1/2}, \quad (5.3)$$

where J is antiunitary and $\Delta := FS$ is the self-adjoint, positive modular operator. The Tomita-Takesaki theorem [11] provides us with a one-parameter group of state dependent isomorphisms α_t^ω on $\mathcal{R}^\omega(\mathcal{O}_s^x)$, defined by

$$\alpha_t^\omega(A) = \Delta^{-it} A \Delta^{it}, \quad \text{for } A \in \mathcal{R}_{\max}^\omega. \quad (5.4)$$

So, as a consequence of commutant duality and isotony assumed above, we obtain here a strongly continuous unitary implementation of the modular group of ω , which is defined by the 1-parameter family of isomorphisms (5.4), given as conjugate action of operators $e^{-it \ln \Delta}$, $t \in \mathbb{R}$. By (5.4) the modular group, for a state ω on the net of von Neumann algebras, defined by $\mathcal{R}_{\max}^\omega$, might be considered as a 1-parameter subgroup of the dynamical group. Note that, with Eq. (5.2), in general, the modular operator Δ is not located on \mathcal{O}_s^x . Therefore, in general, the modular isomorphisms (5.4) are not inner. The modular isomorphisms are known to act as inner isomorphisms, iff the von Neumann algebra $\mathcal{R}^\omega(\mathcal{O}_s^x)$ generated by ω contains only semifinite factors (type I and II), i.e. ω is a semifinite trace.

Above we considered concrete von Neumann algebras $\mathcal{R}^\omega(\mathcal{O}_s^x)$, which are in fact operator representations of an abstract von Neumann algebra \mathcal{R} on a GNS Hilbert space \mathcal{H}^ω w.r.t. a faithful normal state ω . In general, different faithful normal states generate different concrete von Neumann algebras and different modular isomorphism groups of the same abstract von Neumann algebra.

The outer modular isomorphisms form the cohomology group $\text{Out}\mathcal{R} := \text{Aut}\mathcal{R}/\text{Inn}\mathcal{R}$ of modular isomorphisms modulo inner modular isomorphisms. This group is characteristic for the types of factors contained in the von Neumann algebra [16]. Per definition $\text{Out}\mathcal{R}$ is trivial for inner isomorphisms. Factors of type III_1 yield $\text{Out}\mathcal{R} = \mathbb{R}$.

In the case of thermal equilibrium states, corresponding to factors of type III_1 , there is a distinguished 1-parameter group of outer modular isomorphisms, which is a subgroup of the dynamical group.

Looking for a geometric interpretation for this subgroup, parametrized by \mathbb{R} , it should not be a coincidence that our partial order defined above is parametrized by open intervals (namely $]s_{\min}, s_{\max}[$ for the full net and, in the case of (4.7), $]s_z, s_{\max}[$ for the non-Abelian part), and hence diffeomorphically likewise by \mathbb{R} . This way, dilations of the open sets \mathcal{O}_s^x

within the open interval may give a geometrical meaning to the 1-parameter group of outer modular isomorphisms of thermal equilibrium states. Even more, this gives support to the hypothesis[17] of a thermal time. Indeed, a local equilibrium state might be characterized as a KMS state[11, 18] over the algebra of observables on a (suitably defined) double cone, whence the 1-parameter modular group in the KMS condition might be related to the time evolution. Note that, for double cones, a partial order can be related to the split property of the algebras[10].

Now, the details of the isotony condition (3.1), in relation to the modular invariance (5.4), allow rather immediately to draw some further conclusions, which have not yet been spelled out in previous investigations[7, 8]. First assume strict isotony, i.e.

$$s_1 < s_2 \Rightarrow \mathcal{R}^\omega(\mathcal{O}_{s_1}^x) \subsetneq \mathcal{R}^\omega(\mathcal{O}_{s_2}^x) . \quad (5.5)$$

Covariance w.r.t. local dilations then implies isomorphic algebras $\mathcal{R}^\omega(\mathcal{O}_{s_1}^x) \cong \mathcal{R}^\omega(\mathcal{O}_{s_2}^x)$, for $s_{\min} < s_1 < s_2 < s_{\max}$, whence, in particular, all algebras have the same von Neumann type. Obviously, here the condition (4.7) would lead to a totally Abelian net, if the commutant is not taken in a larger algebra $\mathcal{R}_B^\omega \supset \mathcal{R}_{\max}^\omega$. Therefore (4.7) is not further considered here. With the commutant duality (4.5) w.r.t. $\mathcal{R}_{\max}^\omega$ above, $\mathcal{R}_{\min}^\omega$ is Abelian. Hence, the net contains non-Abelian algebras (in particular those with type III₁) only if minimal and maximal sets and algebras are excluded from the net. $\mathcal{R}_{\min}^\omega$ and $\mathcal{R}_{\max}^\omega$ then only exist as inductive limits of a net of isomorphic algebras, while $\partial\mathcal{O}_{\min}^x$ resp. $\partial\mathcal{O}_{\max}^x$ then are horizon-like boundaries of the open manifold supporting the net. Here, only local regions with $s \in]s_{\min}, s_{\max}[$ are admissible for measurement.

Note that in the case where the manifold carries a (pseudo-)metric g , the net must not only be consistent with local dilations of open sets, but also with local dilations of the metric, which take the form of conformal transformations $g(x) \mapsto e^{2\phi(x)}g(x)$, with smooth scale field ϕ on M . If g is consistent with $M =]s_{\min}, s_{\max}[\times \Sigma$ (e.g. by global hyperbolicity), consistency with covariance demands that ϕ is homogeneous on the "horizons", i.e. $\phi(s_{\min}, y) \equiv \phi(s_{\min})$ and $\phi(s_{\max}, y) \equiv \phi(s_{\max})$ for all $y \in \Sigma$.

6 Discussion

It is clear now that (4.7) does not make sense, if the commutant for all algebras of the net is taken w.r.t. $\mathcal{R}_{\max}^\omega$, whence all algebras would be Abelian. Even if we drop (4.7), with this choice of commutant, at least $\mathcal{R}_{\min}^\omega$ is Abelian.

In principle, there remains the possibility, to take the commutant w.r.t. a larger algebra $\mathcal{R}_B^\omega \supset \mathcal{R}_{\max}^\omega$. Then we may obtain also non-Abelian $\mathcal{R}_{\min}^\omega$. However, further examinations are necessary until we can decide clearly which way is the better.

Note however that only the commutant duality (4.5), but not (4.7), was essential for the extraction of the modular group. If we assume von Neumann algebras (with factors) of type III₁, or likewise the existence of local equilibrium states, with the the partial order and related dilations above, it is possible to make natural choices for time and causality.

A thermal time may be obtained from the geometric notion of dilations of the open sets. For any $x \in M$, the parameter s measures the extension of the sets \mathcal{O}_s^x . As accessibility

regions for a local measurement in M , these sets naturally increase with time. Hence it is natural to suggest that the parameter s might be related to a (thermal) time t such that, for any set \mathcal{O}_s^x , $s > s_{\min}$, we have $t < s$ within the set and $t = s$ on the boundary $\partial\mathcal{O}_s^x$.

This thermal time here is related to growing non-identical algebras of increasing support. If covariance is kept as a condition, these algebras are nevertheless isomorphic. A non-isomorphic growth of algebras requires to release the covariance condition. Then, the growth of abstract (isomorphism classes) of algebras may define an arrow of time.

For the ultralocal case ($s_{\min} \rightarrow 0$, without UV cutoff), it is possible to construct the causal structure for a space-time from the corresponding net of operator algebras[19]. Let us consider here the (a priori given) underlying manifold M of the net. Locally around any point $x \in M$ one may induce open double cones as the pullback of the standard double cone which is the conformal model of Minkowski space. These open double cones then carry natural notions of time and causality, which are preserved under dilations. Therefore it seems natural to introduce locally around any $x \in M$ a causal structure and time by specializing the open sets to be open double cones \mathcal{K}_s^x located at x , with time-like extension $2s$ between the ultimate past event p and the ultimate future event q involved in any measurement in \mathcal{K}_s^x at x (time s between p and x , and likewise between x and q). Since the open double cones form a basis for the local topology of M , we might indeed consider equivalently the net of algebras located on open sets

$$\mathcal{O}_s^x := \mathcal{K}_s^x . \quad (6.1)$$

Although some (moderate form of) locality might be indeed an indispensable principle within any reasonable theory of observations, it is nevertheless an important but difficult question, under which consistency conditions a local notion of time and causality might be extended from nonzero local environments of individual points to global regions. This is of course also related to the non-trivial open question, how open neighbourhoods of different points $x_1 \neq x_2$ should be related consistently. Although, in a radical attempt to avoid some part of these difficulties, one might try to replace the notion of points and local regions by more abstract algebraic concepts, a final answer to these questions has not yet been found. At least it seems natural that, on manifolds with no causal relations (like pure space without time), the net should satisfy a disjoint compatibility condition,

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \quad \Rightarrow \quad [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0. \quad (6.2)$$

This condition is e.g. also satisfied for Borchers algebras. Of course the inverse of (6.2) is not true in general.

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References

- [1] C. Rovelli, Loop Quantum Gravity and Black Hole Physics, gr-qc/9608032.
- [2] M. Rainer, *J. Math. Phys.* **35**, 646 (1994).
- [3] K. Fredenhagen K. and R. Haag, *Commun. Math. Phys.* **108**, 91 (1987).
- [4] H. Salehi, *Class. Quant. Grav.* **9**, 2556 (1992).
- [5] R. Haag and D. Kastler, *J. Math. Phys.* **5**, 848 (1964).
- [6] H. Salehi, *Int. J. Theor. Phys.* **36**, 143 (1997).
- [7] M. Rainer, General Regularized Algebraic Nets for General Covariant QFT on Differentiable Manifolds, gr-qc/9705084.
- [8] M. Rainer and H. Salehi, A Regularizing Commutant Duality for a Kinematically Covariant Partial Ordered Net of Observables, in: *Proc. XXI. Int. Sem. on Group Theor. Methods*, Goslar (1996), eds. H. D. Doebner *et al.*; gr-qc/9708059.
- [9] U. Bannier, *Commun. Math. Phys.* **118**, 163 (1988).
- [10] U. Bannier, *Int. J. Theor. Phys.* **33**, 1797 (1994).
- [11] R. Haag, Local Quantum Physics (Springer Verlag, Berlin, 1992).
- [12] S. Doplicher, R. Haag, and J. E. Roberts, *Commun. Math. Phys.* **13**, 1 (1969); *Commun. Math. Phys.* **15**, 173 (1969); *Commun. Math. Phys.* **23**, 199 (1971); *Commun. Math. Phys.* **35**, 49 (1974).
- [13] Wollenberg M. 1990, On Causal Nets of Algebras, pp. 337-344 in: *Operator Theory, Advances and Applications* **43**, *Linear Operators in Function Spaces*, *Proc. 12th Int. Conf. on Operator Theory*, Timișoara, Romania (June 6-16, 1988), eds.: H. Helson, B. Sz.-Nagy, and F.-H. Vasilescu.
- [14] H. Salehi, pp. 357 ff in: *Generalized symmetries in Physics*, eds. H. D. Doebner *et al.* (World Scientific, Singapore, 1994).
- [15] A. Ashtekar and C. Isham, *Class. Quant. Grav.* **9**, 1433 (1992).
- [16] A. Connes, Noncommutative Geometry (Academic Press, N.Y., 1995).
- [17] A. Connes and C. Rovelli, *Class. Quant. Grav.* **11**, 2899 (1994).
- [18] O. Bratteli and D. Robinson, Operator algebras and quantum statistical mechanics I (Springer Verlag, Berlin, 1981).
- [19] M. Keyl, *Rev. Math. Phys.* **8**, 229 (1996).